Endogenous Cycles In Heterogeneous Agent Models: A State-Space Approach

Filippo Gusella[a,b] Giorgio Ricchiuti[a,b]

[a] *Università degli Studi di Firenze*

[b] *Complexity Lab in Economics (CLE), Università Cattolica del Sacro Cuore, Milano*

Abstract

This paper proposes an empirical test to depict possible endogenous cycles within Heterogeneous Agent Models (HAMs). We consider a 2-type HAM into a standard small-scale dynamic asset pricing framework. Fundamentalists base their expectations on the fundamental value, while chartists, subject to self-fulfilling moods, consider the level of past prices. Because these strategies, by their nature, cannot be directly observed but can cause the response of the observed data, we construct a state-space model where agents' beliefs are considered the unobserved state components and from which the heterogeneity of fundamentalist-chartist trader cycles can be mathematically derived and empirically tested. The model is estimated using the S&P500 index for the period 1990-2020 at different time scales, specifically, quarterly, monthly, and daily. We find empirical evidence of endogenous damped fluctuations with a higher percentage of chartists in the short-term horizon. In addition, results indicate time-varying behavioral heterogeneity within-group. Moreover, the model exhibits better long-run out-of-sample forecasting accuracy compared to the benchmark random walk model.

Keywords: Heterogeneous Agent Models, Fundamentalists, Chartists, Endogenous Cycles, State-Space Model

JEL codes: C13, C50, G10, G12, G15, E32

Corresponding author: Giorgio Ricchiuti, Via delle Pandette, 32 - 50127 Firenze, giorgio.ricchiuti@unifi.it

Acknowledgment: An earlier version of this paper was presented at the 27th international conference of CEF, the 24.5th WEHIA workshop, the 42nd annual meeting of the ASSET, the 62nd annual conference of the SIE and the international conference on empirical economics at PSU-Altoona (USA). We are grateful to the participants for their helpful comments and suggestions. We also thank Fabrizio Cipollini, Domenico Delli Gatti, Gabriele Fiorentini, Thomas Lux, Maurizio Motolese, Marco Pangallo and Eleni George Papageorgiou for useful comments, which allowed us to improve the quality of the paper, and Yuri Basile, for suggestions on data. The usual disclaimer applies.

1 Introduction

In his seminal paper, Alan Kirman (1992) shows that the choice of one 'representative' standard utility-maximizing individual 'is not simply an analytical convenience [...], but is both unjustified and leads to conclusions which are usually misleading and often wrong'. At the end of the last century, two lines of research on the role of heterogeneous agent emerged. A first line extends the traditional Macro DSGE literature to heterogeneous agent and it comes back to the works of Bewley (1986), Aiyagari (1994), and Huggett (1993), where a continuum of agents (primarily, but not only, consumers) is introduced in order to discuss uncertainty aggregates. A review of these models is provided in Algan et al. (2014). A second line of research is more focused on the role of (groups of) agents (fundamentalists and chartists for example) in determining complex dynamics of price fluctuations. Hommes (2006) and LeBaron (2006) survey the first papers. While sharing the sensitivity toward the need to abandon the representative agent approach, these two lines of research have developed independently.

Our contribution lies in the second literature; for this reason, when we talk about Heterogeneous Agent Models (HAMs), we will refer to those papers. This behavioral approach undermines two important theoretical pillars: market efficiency based on rational behavior and the homogeneity among investors. The first one is replaced with the idea that economic agents have limited cognitive abilities. The agents, unable to observe all the information about the state of the economy, form their beliefs following mean-reverting and extrapolative expectations feeding the boom and bust in real and financial cycles. The second one is substituted with the idea that heterogeneity among agents is pervasive.

Pioneered by Day and Huang (1990), Chiarella (1992), De Grauwe et al. (1993), Lux (1995) and Brock and Hommes (1998), the HAM theoretical literature is now mature and the results achieved are strong (see as survey LeBaron, 2006; Franke, 2008; Chiarella et al., 2009; Hommes, 2021).[1] On the other hand, there is a growing empirical literature, even if we cannot observe any consensus on the estimation methodology (for recent surveys, refer to Lux and Zwinkels, 2018 and Ter Ellen and Verschoor, 2018). Franke and Westerhoff (2017) note two approaches: direct and indirect. The first method employs surveys to measure the sentiments of a specific group of the population, typically the momentum traders, and thus explain their behavior (see, for example, Jongen et al., 2012 and Goldbaum and Zwinkels, 2014). The second considers a model as a whole and strives to estimate its parameters (Ahrens and Reitz, 2005; Boswijk et al., 2007; Manzan and Westerhoff, 2007; De Jong et al., 2009; Lof, 2012; Chiarella et al., 2012; Goldbaum and Zwinkels, 2014; Frijns and Zwinkels, 2018).

Concerning the latter, we can distinguish between two sub-groups that differ for the inference method (Kukacka and Barunik, 2017). In the first, key structural features of HAMs can be estimated straightaway. Depending on the complexity of the models, we can list - among others - the linear/nonlinear least squares

[1]Over the years, the theoretical literature has applied the concept of heterogeneity in the expectation not only for equity prices but also for the exchange rate market (see, for example, De Grauwe and Grimaldi, 2006, Gori and Ricchiuti, 2018 and Bassi et al., 2023 among others), the housing market (Dieci and Westerhoof, 2013) and to analyze macro-model dynamics (Dosi et al., 2020; Kukacka and Sacht, 2023).

(Boswijk et al., 2007; Hommes and in't Veld, 2017; Manzan and Westerhoff, 2007), the maximum and quasi-maximum likelihood (Alfarano et al., 2006; Chiarella et al., 2014 De Jong et al., 2009 and De Jong et al., 2010; Bolt et al., 2019), the vector autoregression, and vector error correction approaches (Lof, 2015; Frijns and Zwinkels, 2018). These works confirm the theoretical models, recovering the belief coefficients of different groups of speculators. Moreover, price movements in asset prices can derive from a switching mechanism that moves agents among different rules. In the second method, estimation based on simulating artificial data from the model is instead used (Franke, 2009; Franke and Westerhoff, 2011; Chen and Lux, 2018): through the method of simulated moments, it is possible to depict phenomena which are the consequence of behavioral biases, such as volatility clustering and long-memory effects (Schmitt, 2021). These works show that sentiment dynamics is pivotal in explaining stylized facts detected in financial time series and in replicating observed anomalies within these markets.[2]

Notwithstanding the different estimation methods, there is agreement on a fundamental pillar: the heterogeneity of the expectations between different groups of agents, which can influence price dynamics. These heuristic beliefs remain observable under standard econometric approach, meaning that previous estimation methods were applied for parameter estimation but did not provide an avenue for filtering information on unobserved states. In fact, econometric analyses are performed by combining observed and unobserved variables into a single or multiple observed price dynamics. However, because behavioral heuristics are unobservable variables, unlike the papers mentioned above, the present paper, using data on S&P500 for the period 1990-2020, employs a state-space model formulation to estimate possible endogenous cycles that emerge as a consequence of latent speculative behavior. A state-space representation is defined by a mathematical model of a system composed of observed output and unobserved state variables (Durbin and Koopman, 2012). State variables evolve over time, influencing outputs that depend on the unobserved dynamics. In our case, the hidden speculative beliefs influence the observed price dynamics, possibly generating endogenous instability phenomena in the form of endogenous fluctuations. Besides the tractability of the model, the main advantage of this framework is that, through filtering information on unobserved states, the researcher is able to test whether behavioral rules lead to cyclical dynamics in the observed asset prices.

To perform it, we proceed as follows: We assume that mean-variance utility optimizer investors populate the economy. From this scenario, we first set up the heterogeneous agent model setting.[3] Consistent with HAMs, the diversity of agents is reflected in their expectation formation. Specifically, financial traders employ

[2]The papers mentioned above refer primarily to the heterogeneous agent models where heterogeneity concerns groups - specifically, the strategies adopted by groups. It is worth underlining that a great effort also concerns the econometric validation of agent-based models, which we would call *pure* ABMs (see Dosi and Roventini, 2019 for a comprehensive review), where heterogeneity is pervasive, given that it is at the level of the single agent (Fagiolo et al., 2019; Monti et al., 2022). As well highlighted by LeBaron and Tesfatsion (2008) and by Grazzini and Richiardi (2015), the estimation of pure agent models is complicated by the heterogeneity of the agents but also by the difficulty of aggregating their actions. Recently, Delli Gatti and Grazzini (2020) proposed both a Bayesian procedure to estimate the parameters of a pure agent model and a method to aggregate the data obtained from the simulations so as to obtain historical series to be used in the model validation phase. See also Lamperti et al., (2018) for ABM estimation via modern machine learning techniques.

[3]In order to help the reader, the diagram in Fig. 1 illustrates the relationship between the classical heterogeneous agent setting with the state-space model.

either fundamental or technical expectation rules in order to predict future prices. Subsequently, to depict the possible endogenous cycles, we consider the agents' beliefs as unobserved state components from which, through a state-space model formulation, the endogeneity of fundamentalist-trend follower trader cycles can be mathematically derived. More precisely, eigenvalues analysis can be performed to study the conditions for oscillations in our discrete dynamic system associated with the unobserved belief components. Once we obtain the cyclical conditions, maximum likelihood estimation via the Kalman filter over the state-space model is performed. At the same time, Monte Carlo simulation analysis is performed as a robustness test. In this way, we can evaluate the presence of endogenous cycles directly from the data and investigate whether they are statistically significant or not. Finally, we estimate the relative shares of the two economic agents in the market with the possibility of recovering the reaction coefficient associated with the speculative expectation rule.

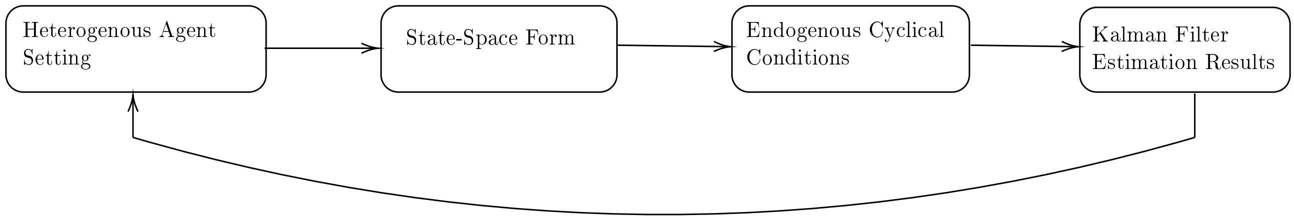

Figure 1: From the heterogeneous agent setting to the state-space model, and back.

Although the state-space model has been around macroeconomics for a while, this instrument only recently has been discovered to be useful for HAMs estimation.[4] Lux (2018; 2021) uses Particle Markov Chain Monte Carlo (PMCMC) and Adaptive PMCMC methods for state-space models to compare two different HAMs. He emphasizes that this methodology could be optimal for HAMs precisely because they typically have latent components and observable variables. Along the same line, Gusella and Stockhammer (2021) propose an empirical test for Minskyan financial cycles in equity and housing prices.

Our paper departs from Gusella and Stockhammer (2021) to depict possible endogenous cycles but highlighting the heterogeneity among agents and seeking to empirically identify different evaluation behaviors. On the one hand, we do not consider an unobserved fundamental price. In our case, fundamentalists base their expectations on the deviation of observed fundamental value from market price, expecting a convergence between them: the fundamental value is identified through the Gordon model (1959), as in Chiarella et al. (2012). On the other hand, chartists, subject to self-fulfilling moods, consider the level of past prices acting as trend followers.[5] Additionally, believing that models may not explain series trends for all time scales, we focus on implementing our model at different time scales, specifically quarterly, monthly, and daily. Often, in theoretical models little attention is paid to what time is and those who did empirical studies used different time scales, but never compared inside the same work.

Differently from the Bayesian approach of Lux (2021), we implement a frequentist approach. Compared to Lux (2018), our quotas are fixed and we consider the strategies adopted by the agents as unobservable. While

[4] It is worth noting that within the literature à la Krusell and Smith (1998), the state-space model approaches and heterogeneity are used from the start.

[5] Throughout the entire paper, we will use the terms chartists, speculators and trend followers interchangeably.

we acknowledge that the assumption of fixed quotas is a strong one, it is noteworthy that this hypothesis has historical precedence in the literature, with one of the seminal articles in the field of HAMs adopting a similar premise (see Day and Huang, 1990).[6] These fixed quotas can also be interpreted as estimates of the shares of chartists and fundamentalists within our sample or as the probabilities that agents adopt one of these two strategies within the defined time span. In that sense, this method allows us to measure and compare how the percentage of chartists and their associated reaction parameters change over different frequency times, making a structural break analysis. Moreover, this choice allows us to construct a reduced-form model for estimation with a feasible search for the mathematical conditions of possible complex eigenvalues. On this point, rather than comprehending the movements of the proportions during a local bubble phenomenon, we are interested in understanding whether endogenous cyclical phenomena are formed without imposing restrictions on the nature of the eigenvalues associated with the parameters of the unobserved beliefs of the agents. In fact, differently from the standard general equilibrium model where the equilibrium is assumed, in the present work, there are no top-down constraints that would prevent the economy from fluctuating out-of-equilibrium. It should also be noted that, as shown by Lux (2021b), most of the empirical nonlinear models do not perform better than a linear chartist-fundamentalist model, and often their fit is worse than the fit of the linear benchmark. Notwithstanding, to overcome the limitation of fixed quotas with the proposed estimation strategy, a state-space rolling window analysis is performed to depict the possible evolution of agents between the groups.

In all the frequencies considered, our results support endogenous cycles, with the modulus of complex eigenvalues lower than one. The percentage of chartists tends to increase in the correspondence of an increase of frequency-time and, as highlighted by the rolling window exercises, for a certain specific period in the sample: before the dot.com crisis and the global financial crisis. At the same time, the intensity of the reaction parameter decreases passing from quarterly to monthly and daily data.

In the light of these considerations, our paper makes four contributions to the empirical behavioral literature on asset pricing with heterogeneous expectations. Firstly, compared to prior estimations of two-type HAMs applied to the S&P 500 index, state-space model has not been thoroughly investigated for searching endogenous cycles as a consequence of unobserved behavioral heuristics. This is where our paper comes in: assuming that unobservable variables drive the dynamics of the observable time series, we directly test if there is a cyclical dynamic in the observed asset prices. As a second novelty, as stressed before, we make a comparison between different time scales. At the moment, comparisons are between different asset prices (Ter Ellen et al., 2021; Gusella and Stockhammer, 2021) or different models on the same asset prices; for example, Lux (2021b) compares the performance and the explanatory power of different HAMs on S&P 500 distortion. Third, because the linearity of the model does not allow us to depict possible local instability phenomena (Beaudry et al., 2017), we run a state-space rolling window analysis to test the evolution of quotas of traders and to see if the speculative position dominates during a specific period of time.

Finally, despite its importance, forecasting procedures are still little explored in this literature. Empirical works within HAMs tradition have focused more on finding heterogeneity in data, while few works, especially

[6] See also Beja and Goldman (1980) for the behavioral finance literature.

for stock market prices, have devoted attention to the possibility of extrapolating forecasting power through the models used (see, for example, Chiarella et al., 2012; Lof, 2012, and Recchioni et al., 2015).[7] Furthermore, state-space models have yet to be used to study the forecasting performance of heterogeneous agent models. As a consequence of this, the last methodological contribution is to evaluate the out-of-sample forecasting power of a state-space HAM compared to the random walk benchmark. With this respect, our analysis reveals that the forecasting performances of RW and behavioral specifications are different in the short-run and in the long-run horizons.

The rest of the paper is organized as follows. Section (2) discusses the theoretical and empirical methodology. Firstly, we set up the heterogeneous agent model in (2.1). The model is then converted into a state-space form (2.2), from which the conditions for endogenous cycles can be mathematically derived (2.3) and the estimation strategy implemented (2.4). Main results are reported in section (3). Specifically, we first show results obtained at quarterly frequency (3.1), later at monthly (3.2), and finally at daily (3.3). After the comparison between the different specifications (3.4), section (4) presents the state-space rolling window analysis, and section (5) compares the out-of-sample predictability of the behavioral specification with that of the random walk hypothesis. Finally, section (6) reports conclusions and final considerations.

2 Methodology

This section describes the proposed estimation strategy to study the presence of endogenous financial fluctuations due to the unobserved beliefs of agents in the financial asset market.

2.1 *The model*

In line with, N financial operators at time t can invest in two types of asset: a risk-free asset that pays a risk-free interest rate r and a single risky asset with price p, paying a stochastic dividend y (Brock and Hommes, 1998).

The dynamics of wealth for investor $i \in N$ at the end of period $t+1$ is described by the following dynamic equation:

$$W_{t+1}^{i} = (1+r)W_t^i + \left[p_{t+1} + y_{t+1} - (1+r)p_t\right] Z_t^i,$$

where W indicates the wealth of agent i while Z is the demand for the risky asset. Investors are mean-variance utility optimizers and their utility can be represented by the following equation:

$$U\left(W_{t+1}^i\right) = E_t^i W_{t+1}^i - (a/2)\, V_t^i\left(W_{t+1}^i\right),$$

where E_t is the expectation operator, V_t is the conditional variance of the wealth of agent i, and $a > 0$ represents the risk aversion parameter, the same for all the agents. The demand for the risky asset is derived from the

[7] For the exchange rate, we can mention Manzan and Westerhoff (2007), De Jong et al. (2010) and Jongen et al. (2012).

following maximization process:

$$\max_{z}^{i}\left\{E_t^i W_{t+1}^i - (a/2)\, V_t^i\left(W_{t+1}^i\right)\right\}.$$

Deriving with respect Z_t and knowing that the variance of wealth is equal to $Var\,(W) = \mathrm{E}\left[(W - \mathrm{E}\,[W])^2\right]$, we obtain:

$$E_t^i\left(p_{t+1}\right) + E_t^i\left(y_{t+1}\right) - (1+r)p_t - Z_t^i a V_t^i\left(p_{t+1} + y_{t+1}\right) = 0.$$

The optimal demand for risky asset reads as follows:

$$Z_t^i = \frac{E_t^i\left(p_{t+1}\right) + E_t^i\left(y_{t+1}\right) - (1+r)p_t}{aV_t^i\left(p_{t+1} + y_{t+1}\right)}.$$

As in Brock and Hommes (1998), for analytical tractability, variance is constant:

$$V_t^i\left(p_{t+1} + y_{t+1}\right) = \sigma^2 \quad \forall\, i \in\, N,$$

and expectations about dividends are homogeneous for all trader types and equal to the conditional expectation:

$$E_t^i\left(y_{t+1}\right) = \bar{y} \quad \forall\, i \in\, N.$$

In the light of this, denoting with $E_t\left(p_{t+1}\right)$ the investors' average expectation about the price of the risky asset at time $t+1$ ($\Sigma_{i=1}^N \frac{E_t^i(p_{t+1})}{N}$), the aggregate demand for the risky asset Z_t reads as follow:

$$Z_t = \Sigma_{i=1}^N Z_t^i = N\frac{E_t\left(p_{t+1}\right) + \bar{y} - (1+r)\, p_t}{a\sigma^2}.$$

Without loss of generality, the outside supply S of the risky asset is equal to zero ($NS = 0$), such that, from the equilibrium condition:

$$\frac{E_t\left(p_{t+1}\right) + \bar{y} - (1+r)p_t}{a\sigma^2} = 0,$$

i.e.:

$$(1+r)p_t = E_t\left(p_{t+1}\right) + \bar{y}.$$

In line with HAMs, heterogeneity of agents is in the expectation formation. Financial traders use a fundamental or a technical expectation rule to forecast future prices. Indicating with δ the market shares of fundamentalists (f), and with $1-\delta$ the market shares of chartists (s) following the technical rule, investors' average price expectations can be formalized in the following way:

$$E_t\left(p_{t+1}\right) = \delta E_t^f\left(p_{t+1}\right) + (1-\delta)\, E_t^s\left(p_{t+1}\right).$$

For the purpose of estimation, it is assumed that both the risk-free rate and the constant expectation about dividends are equal to zero. The price of the risky asset p_t is then expressed as follows:

$$p_t = \delta E_t^f \left(p_{t+1}\right) + \left(1-\delta\right) E_t^s \left(p_{t+1}\right). \tag{1}$$

We now formalize the expectation formation of the two groups of agents considered. Let us start considering the fundamentalists. They believe in the efficient market theory, expecting the price to be equal to the fundamental value p^f. In the HAM literature, the fundamental value is defined through the Gordon growth model (Gordon, 1959). So the fundamentalists' expectation can be defined as:

$$E_t^f \left(p_{t+1}\right) = p_t^f, \tag{2}$$

where E_t^f is the forecast made in period t by the fundamentalists.

As to the chartists, we define their expectation in the following way:

$$E_t^s \left(p_{t+1}\right) = p_t^f + \beta \left(p_{t-1} - p_{t-2}\right), \qquad with \quad \beta > 0 \tag{3}$$

where E_t^s is the forecast made by chartists and β is the reaction coefficient expressing the degree by which they extrapolate the past change in the asset market. At time t, only prices determined at the end of period $t-1$ are known by the agents. Based on the information at time $t-1$, they build their forecasts for future prices at $t+1$.[8]

Substituting Eqs. 2 and 3 in Eq. 1, we obtain:

$$p_t = \delta \left(p_t^f\right) + \left(1-\delta\right) \left(p_t^f + \beta \left(p_{t-1} - p_{t-2}\right)\right),$$

from which:

$$p_t = p_t^f + \left(1-\delta\right) \beta \left(p_{t-1} - p_{t-2}\right).$$

This last equation, with respect to the belief function of chartists B^s, can be rewritten in the following way:

$$p_t = p_t^f + \left(1-\delta\right) B_t^s, \tag{4}$$

with

$$B_t^s = \beta \left(p_{t-1} - p_{t-2}\right).$$

[8] Unlike what is typically done in the HAMs literature, where the "base price" is the price at the previous time, in our case, they take into account the fundamental value in forming expectations before proceeding to speculate based on past prices. This modification allows us to obtain a non-singular transition matrix in the transformation to a state-space model. Please refer to paragraph 2.2 and Franke (2008) for more details.

2.2 *The state-space form*

As agents' behavioral beliefs are unobserved variables, we can construct our state-space model in the context of the unobserved components model. With this modeling strategy, we can reveal the nature and the cause of the dynamic movement of observed variables in an effective way. Indeed, with a state-space model, it is possible to explain an observed variable's behavior by examining the unobserved components' internal dynamic properties. In other words, we can analyze the effect of the unobserved components variable (heuristic beliefs) on the observed variable (asset prices). To perform it, Eq. 4 is substituted in the belief function of the chartists, so to obtain:

$$B_t^s = \beta\left(p_{t-1}^f + (1-\delta)\, B_{t-1}^s - p_{t-2}^f - (1-\delta)\, B_{t-2}^s\right).$$

In line with Franke (2008) and Lux (2021a), the fundamental value is assumed to follow a Brownian motion with increments $\varepsilon_t \sim \mathrm{N}(0, \sigma_\varepsilon)$.[9] We thus have:

$$B_t^s = \beta(1-\delta)\, B_{t-1}^s - \beta(1-\delta)\, B_{t-2}^s + \varepsilon_t.$$

This last equation is the latent equation while Eq. 4 is the observed equation. An essential feature of any state-space model is that the state equation must be a first-order stochastic difference equation (Enders, 2016) so that, in a stochastic matrix-vector formulation, the state-space form assumes the following form:

$$p_t = p_t^f + \begin{bmatrix} (1-\delta) & 0 \end{bmatrix} \begin{bmatrix} B_t^s \\ B_{t-1}^s \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} B_t^s \\ B_{t-1}^s \end{bmatrix} = \begin{pmatrix} \beta(1-\delta) & -\beta(1-\delta) \\ 1 & 0 \end{pmatrix} \begin{bmatrix} B_{t-1}^s \\ B_{t-2}^s \end{bmatrix} + \begin{bmatrix} \varepsilon_t \\ 0 \end{bmatrix}, \tag{6}$$

where ϵ_t is the individual disturbance term normally distributed with mean zero and variance σ_φ^2.

Eq. 5 is the measurement equation, while Eq. 6 is the transition matrix containing the dynamics of the belief function of the group of chartists. The system's dynamics is given by the transition equation, which describes the evolution of the vector of unknown latent variables. Eigenvalues analysis can be performed to study the conditions for oscillations in our discrete dynamic system associated with the unobserved belief component.

2.3 *The cyclical conditions*

Solving for the determinant of the characteristic equation, the roots are the following:[10]

$$\lambda_{1,2} = \frac{\beta(1-\delta) \pm \sqrt{(\beta(1-\delta))^2 - 4\beta(1-\delta)}}{2}.$$

[9] This hypothesis is econometrically tested on the Gordon fundamental time series. Due to space reasons, we do not display here the results. They are available in Supplementary Material A.

[10] In Appendix A, the mathematical steps to obtain eigenvalue condition for cyclical behavior is reported.

In order to have an oscillating behavior, the two eigenvalues have to be complex, so we require:

$$\Delta = (\beta (1-\delta))^2 - 4\beta (1-\delta) < 0,$$

i.e.:

$$\beta (1-\delta) [\beta (1-\delta) - 4] < 0. \tag{7}$$

From the previous equation, we observe that the necessary and sufficient condition to obtain complex eigenvalues reads as follow:

$$0 < (1-\delta) \beta < 4.$$

This mathematical result reflects an important economic condition of the HAMs literature: to obtain endogenous fluctuation phenomena in the system, the percentage of chartists must not be zero ($1-\delta \neq 0$). At the same time, the reaction parameter must be positive ($\beta > 0$). If $\beta = 0$ or $1-\delta = 0$, in line with the efficient market hypothesis, the price will reflect the exogenous fundamental price in an equilibrium condition, which can be affected only by unexpected exogenous shocks. In other words, the model boils down to a benchmark model with fundamental agents having rational and perfect information about the "state" of the economy.

If condition 7 holds, the two associated complex eigenvalues assume the following form:

$$\lambda_{1,2} = \frac{Tr(A)}{2} \pm i \frac{\sqrt{-\Delta}}{2} = a \pm ib, \quad with \quad A = \begin{pmatrix} \beta (1-\delta) & -\beta (1-\delta) \\ 1 & 0 \end{pmatrix}.$$

In an equivalent trigonometric form:

$$\lambda_{1,2} = \rho (\cos\theta \pm i \sin\theta),$$

where the modulus is:

$$\rho = \left(a^2 + b^2\right)^{1/2},$$

and respect to our parameters of interest:

$$\rho = \sqrt{\frac{(\beta (1-\delta))^2}{4} + \frac{-(\beta (1-\delta))^2 + 4 (\beta (1-\delta))}{4}}.$$

If $\sqrt{\beta (1-\delta)} = 1$, we obtain constant-amplitude endogenous fluctuations. With the modulus of eigenvalue lower than one ($\sqrt{\beta (1-\delta)} < 1$), we observe damped endogenous fluctuations, while explosive endogenous fluctuations if $\sqrt{\beta (1-\delta)} > 1$.

For simplicity on notations, we set

$$\begin{cases} \beta(1-\delta) = a_{11} \\ -\beta(1-\delta) = a_{12} \end{cases} \tag{8}$$

Summing up, the state-space model can be rewritten in compact form as:

$$p_t = p_t^f + HB_t,$$

$$B_t = AB_{t-1} + \varphi_t \qquad \varphi_t \sim \mathrm{N}(0, Q),$$

where p_t is the observable asset price,

$$B_t = \begin{bmatrix} B_t^s \\ B_{t-1}^s \end{bmatrix},$$

is the state vector,

$$H = \begin{bmatrix} (1-\delta) & 0 \end{bmatrix},$$

is the measurement matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ 1 & 0 \end{pmatrix},$$

is the transition matrix and φ_t is the vector containing the state disturbance of the unobserved component, normally distributed with mean zero and variances collected in the diagonal matrix Q.

2.4 *Estimation Strategy*

After converting the structural model into reduced form for estimation purposes, we proceed to estimate it. We use S&P 500 data, employing the closing adjusted values from January 1990 to December 2020, with a quarterly, monthly and daily frequency. As part of the econometric analysis, all series are in log level. The fundamental price is calculated through the Gordon growth model as in Chiarella et al. (2012). Defining d_t as the dividend flow, g the average growth rate of dividends, and r the average required return, the fundamental value of asset price can be defined as:

$$p_t^f = u_t = d_t \frac{(1+g)}{(r-g)}.$$

Following Chiarella et al. (2012), we assume that r is equal to the sum of the average dividend yield $\tilde{y}$ and the average rate of capital gain $\tilde{x}$. The Gordon growth model implies that $\tilde{x}$ is equal to g, so as to obtain:

$$u_t = d_t \frac{(1+g)}{\tilde{y}}. \tag{9}$$

From Eq. 9, the fundamental value is now equal to the current dividend flow multiplied by a constant multiplier.

After we obtain the fundamental value and the model is in a state-space form, the cyclical parameters a_{11} and a_{12} with the proportion of chartist traders $(1-\delta)$ are estimated by maximum likelihood estimation via the Kalman Filter. More precisely, the parameters of the model are estimated by maximum likelihood using the prediction error decomposition approach, where the one-step prediction and updating equations are calculated via the state-space form using the Kalman filter.[11] This filter is an algorithm that uses a data series of an observed variable to produce estimates of unknown variables that generate the dynamics of the observed variable. Once the iterative Kalman filter algorithm extracts the unobserved states by performing forward recursion over the state-space model (Harvey, 1989), we can check from the data if the cyclical fluctuations are an endogenous outcome that we find as a result of the interaction of the two unobserved components (i.e., if the cyclical condition $\beta(1-\delta)[\beta(1-\delta)-4]<0$ is respected). Moreover, after estimating cyclical parameters (a_{11} and a_{12}) with the proportions of trend followers $(1-\delta)$, from Eq. 8 we can obtain the value of reaction parameter for chartists (β).

In the optimization process for maximum likelihood estimation, we assume that $1-\delta$ is between zero and one. Moreover, the equality constrain $a_{11}+a_{12}=\beta(1-\delta)-\beta(1-\delta)=0$ is assumed so as to obtain only one positive value of the reaction coefficient β.

3 Estimation Results

The results are reported as follows. We first analyze the model at quarterly frequency and then at monthly and daily frequency. Finally, as a novelty, we compare the different specifications.

3.1 *Quarterly Results*

Table 1 shows the obtained results for time series at quarterly frequency. As it turns out, the signs of a_{11} and a_{12} respect condition in Eq. 7 for oscillatory phenomena. The parameters are within the range size $\{\beta(1-\delta)[\beta(1-\delta)-4]<0\}$ and estimates are statistically significant at the ten percent level. In particular, we have damped fluctuations ($\sqrt{\beta(1-\delta)}<1$) with the modulus equal to 0.65.

Looking at the percentage of the two different types of agents, we notice that chartists $(1-\delta)$ are the minority in comparison with fundamentalists (δ). More precisely, 65% of the agents are estimated to be fundamentalists while 35% are chartists; it is worth noting that the percentage of chartists is statistically significant at the one percent level. Once we obtain these estimates, from Eq. (8) it is possible to recover the value of β, which is equal to 1.2.

[11]See appendix B for the mathematical steps of the filtering procedure.

Table 1: Estimation Results [S&P500 Quarterly Data]

	Cyclical Parameters			
	a_{11}	a_{12}	a_{21}	a_{22}
Values	0.42* (0.26082)	-0.42* (0.26082)	1	0
	Percentage and Reaction Coefficients			
	$1-\delta$	β		
Values	0.35*** (0.01122)	1.2		
	State Disturbance			
	σ_ε			
Values	0.89080*** (0.02581)			
	Cyclical Conditions			
$\beta(1-\delta)[\beta(1-\delta)-4]<0$	*Fulfilled*			
$\left(a^2+b^2\right)^{1/2}$	*Damped*			
	Info Model			
Eigenvalues	-0.2102 ± 0.6134i			
Modulus	0.65			
Log-likelihood	-31.7849			
Akaike Info Criterion	71.5697			
Bayesian Info Criterion	82.8184			

Notes: Standard errors in parentheses.
*, **, *** denotes statistical significance at the 10%, 5%, and 1% levels respectively.

Overall, these results provide empirical support for the existence of endogenous financial cycles in equity prices as a consequence of the different expectation rules defined in our model. Moreover, the previous findings hold also when we implement Monte Carlo simulation test. We simulate $n = 1000$ sample path of observations from the estimated model, randomly generating state disturbances from the standard normal distribution by plugging them into the state-space model. Once we obtain the results, we consider the mean value of the 1000 cyclical conditions to see if the endogenous fluctuations are satisfied. As previously found, results support the hypothesis of endogenous cyclical condition (see Table 2).[12]

[12]The estimated model is evaluated with diagnostic checks on residuals. In state-space models, these tests are applied to what are known as the standardized prediction errors, which are the ratio between the one-step ahead prediction errors obtained from the Kalman filter procedure and its variance. Residuals should satisfy the following three properties (Commandeur and Koopman, 2007): independence, homoscedasticity, and normality. The assumption of independence of the residuals is examined with the sample auto-correlation function, homoscedasticity is checked with the Engle ARCH test to evaluate the null hypothesis of no autoregressive conditional heteroscedasticity effects, and finally, the Jarque-Bera test is used to check that residuals are normally distributed. Due to space reasons, we do not display here the results. They are available in Supplementary Material B.

Table 2: Monte Carlo Simulation Results [S&P500 Quarterly Data]

	Cyclical Parameters			
	a_{11}	a_{12}	a_{21}	a_{22}
Values	0.4128 [0.2830 - 0.5430]	-0.4128 [0.2830 - 0.5430]	1	0
	Percentage and Reaction Coefficients			
	$1-\delta$	β		
Values	0.41 [0.32 - 0.49]	1		
	State Disturbance			
	σ_ε			
Values	0.7609 [0.6019 - 0.9199]			
	Cyclical Conditions			
$\beta\left(1-\delta\right)\left[\beta\left(1-\delta\right)-4\right]<0$		*Fulfilled*		
$\left(a^2+b^2\right)^{1/2}$		*Damped*		
	Info Model			
Eigenvalues	0.2064 ± 0.6084i			
Modulus	0.64			

Notes: Confidence interval in squared brackets.

The values of parameters are the mean value of 1000 MC simulation results.

3.2 *Monthly Results*

The estimates of our monthly model are displayed in Table 3. Starting with the maximum likelihood estimates, we notice that the cyclical parameters are highly significant, with estimated coefficients significant at the one percent statistical level. Moreover, a_{11} and a_{12} respect the mathematical condition in Eq. 7 $\{\beta(1-\delta)[\beta(1-\delta)-4]<0\}$ for damped fluctuations $[\sqrt{\beta(1-\delta)}<1]$. As before, these results confirm the presence of endogenous fluctuation with the modulus of complex eigenvalues near one ($\rho = 0.73$), likely to generate persistent fluctuations.

Table 3: Estimation Results [S&P500 Monthly Data]

	Cyclical Parameters			
	a_{11}	a_{12}	a_{21}	a_{22}
Values	0.53*** (0.23568)	-0.53*** (0.23568)	1	0
	Percentage and Reaction Coefficients			
	$1-\delta$	β		
Values	0.63*** (0.01004)	0.85		
	State Disturbance			
	σ_ε			
Values	0.48935*** (0.00783)			
	Cyclical Conditions			
$\beta(1-\delta)[\beta(1-\delta)-4]<0$	*Fulfilled*			
$\left(a^2+b^2\right)^{1/2}$	*Damped*			
	Info Model			
Eigenvalues	-0.2688 ± 0.6822i			
Modulus	0.73			
Log-likelihood	-89.2338			
Akaike Info Criterion	186.468			
Bayesian Info Criterion	202.143			

Notes: Standard errors in parentheses.
*, **, *** denotes statistical significance at the 10%, 5%, and 1% levels respectively.

On the basis of monthly data, now chartists ($1-\delta = 63\%$) are in the majority with respect to fundamentalists ($\delta = 37\%$). This effect is compensated by the intensity of the reaction parameters. We observe a value for β equal to 0.85. In this sense, with quarterly data, the speculative position is primarily taken from beliefs that are not shared by the majority of the market with a high reaction coefficient ($\beta = 1.2$). The opposite for monthly data, where the speculative position is primarily taken from beliefs shared by the majority of the market with a lower reaction coefficient ($\beta = 0.85$).

Again, as we have done for quarterly data, our results are enriched with Monte Carlo simulation results. Results are summarized in Table 4. Overall, we can conclude that estimation results provide empirical support for endogenous cycles at monthly frequency.

Table 4: Monte Carlo Simulation Results [S&P500 Monthly Data]

	Cyclical Parameters			
	a_{11}	a_{12}	a_{21}	a_{22}
Values	0.5350 [0.4639 - 0.6061]	-0.5350 [0.4639 - 0.6061]	1	0
	Percentage and Reaction Coefficients			
	$1-\delta$	β		
Values	0.51 [0.1873 - 0.8395]	0.88		
	State Disturbance			
	σ_ε			
Values	0.6832 [0.1120 - 1.2544]			
	Cyclical Conditions			
$\beta(1-\delta)[\beta(1-\delta)-4] < 0$		*Fulfilled*		
$(a^2+b^2)^{1/2}$		*Damped*		
	Info Model			
Eigenvalues	$-0.2675 \pm 0.6808i$			
Modulus	0.7315			

Notes: Confidence interval in squared brackets.
The values of parameters are the mean value of 1000 MC simulation results.

3.3 *Daily Results*

The results for time series at daily frequency are presented in Table 5. Because no data on dividends at daily frequency are available, we interpolate Gordon data to obtain a larger number of time series observations. In particular, we increase the frequency of observation passing from monthly to daily frequency.

Comparing these results with previous ones, we notice important similarities and differences. Concerning the cyclical conditions, as for quarterly data, these are formally satisfied $\{\beta(1-\delta)[\beta(1-\delta)-4]<0\}$ with damped fluctuations at ten percent statistical significance $[\sqrt{\beta(1-\delta)}<1]$. As for monthly data, we find a higher proportion of chartists than fundamentalists; 71% of agents are estimated to be chartists at one percent statistical evidence, and the remaining 29% are fundamentalists. Again we can obtain the extent of chartists extrapolation coefficient from Eq. 8, which is 0.60. Overall, as shown in Table 6, the previous findings hold also when we implement Monte Carlo simulation test.

Table 5: Estimation Results [S&P500 Daily Data]

	Cyclical Parameters			
	a_{11}	a_{12}	a_{21}	a_{22}
Values	0.43* (0.24365)	-0.43* (0.24365)	1	0
	Percentage and Reaction Coefficient			
	$1-\delta$	β		
Values	0.71*** (0.00252)	0.60		
	State Disturbance			
	σ_{ε}			
Values	0.00151*** (0.00783)			
	Cyclical Conditions			
$\beta(1-\delta)[\beta(1-\delta)-4]<0$		*Fulfilled*		
$(a^2+b^2)^{1/2}$		*Damped*		
	Info Model			
Eigenvalues	$-0.2165 \pm 0.6214i$			
Modulus	0.66			
Log-likelihood	-1890.21			
Akaike Info Criterion	3788.42			
Bayesian Info Criterion	3816.28			

Notes: Standard errors in parentheses.
*, **, *** denotes statistical significance at the 10%, 5%, and 1% levels respectively.

Table 6: Monte Carlo Simulation Results [S&P500 Daily Data]

	Cyclical Parameters			
	a_{11}	a_{12}	a_{21}	a_{22}
Values	0.4227 [0.4064 - 0.4390]	-0.4227 [0.4064 - 0.4390]	1	0
	Percentage and Reaction Coefficients			
	$1-\delta$	β		
Values	0.76 [0.5635 - 0.9565]	0.55		
	State Disturbance			
	σ_{ε}			
Values	0.4116 [0.2961 - 0.5271]			
	Cyclical Conditions			
$\beta(1-\delta)[\beta(1-\delta)-4]<0$		$Fulfilled$		
$\left(a^2+b^2\right)^{1/2}$		*Damped*		
	Info Model			
Eigenvalues	-0.2114 ± 0.6148i			
Modulus	0.6501			

Notes: Confidence interval in squared brackets.
The values of parameters are the mean value of 1000 MC simulation results.

3.4 *From Quarterly to Daily Frequency*

At this point of the analysis, we can compare and comment the obtained results. First, as we can see from Table 7, the percentage of fundamentalists is higher than the percentage of chartists at quarterly frequency. A similar phenomenon can be detected in the first specification of the Recchioni et al. (2015) model, where the fundamentalist strategy remains predominant over the chartist one. However, with an increase in frequency-time, the percentage of speculators $(1-\delta)$ tends to increase: $(1-\delta)$ equals 0.35 with quarterly data, 0.63 at monthly frequency, and 0.71 with daily data. These results align with HAM literature, which presupposes an increasing speculative position in the short run period. At the same time, our results are in line with Poterba and Summers (1988) and Frankel and Froot (1990). In a different theoretical and empirical context, Poterba and Summers (1988) characterize the different dynamics of asset returns for the short vs. long horizon time, suggesting that stock returns exhibit positive serial correlation over short periods and negative over longer intervals. Such differences could be the consequence of the presence of different agents which coexist in the market: the mean-reverting traders and noise traders/feedback traders (see also Cutler et al., 1990). Both groups of traders affect prices, and the introduction of speculative traders appears to be a plausible explanation for transitory price dynamics in the short horizon.

Table 7: Comparison between the different frequencies

		Behavioral HAM		
		Quarterly	*Monthly*	*Daily*
Pct. of fundamentalists	δ	0.65	0.37	0.29
Pct. of chartists	$1-\delta$	0.35	0.63	0.71
Spec. reaction coeff.	β	1.2	0.88	0.60
Akaike info criterion	AIC	71.5697	186.468	3788.42
Bayesian info criterion	BIC	82.8184	202.143	3816.28
Modulus	ρ	0.65	0.73	0.66
Cycles		*yes*	*yes*	*yes*

Second, the reaction coefficient β is equal to 1.2 with quarterly data, 0.88 at monthly frequency, and 0.60 with daily data. In other words, the intensity of the reaction parameter decreases from quarterly to monthly data and daily data. This outcome, which we do not find in any other paper within the literature, holds logical significance. In fact, economic agents extrapolate data more intensely at intervals of low frequency compared to intervals of high frequency, expecting, in the latter case, smaller quantitative changes.

Finally, we observe a tendency towards stable oscillatory patterns for all the different frequency specifications: ρ equals 0.65 with quarterly data, $\rho = 0.73$ at monthly frequency, and $\rho = 0.66$ with daily data. The system stays within a stable region and does not generate divergent trajectories. Because linearity biases the results toward damped fluctuations (Beaudry et al., 2017), nonlinearity would enrich the possible trajectories highlighting the destabilizing tendency of chartist behavior and the stabilizing force exerted by fundamentalists. However, in our opinion, this limitation is overcome by three crucial aspects. Firstly, HAMs are disequilibrium models; unlike what happens in the standard general equilibrium model where the equilibrium is posited as an assumption, in the present work, there is no top-down restriction preventing the economy from fluctuating out-of-equilibrium. The dynamics emerge as an endogenous outcome generated by the interactions among the agents (see also Delli Gatti et al., 2018). In this sense, we do not assume the existence of cyclical phenomena, but also the existence of damped phenomena is not assumed: in fact, we do not impose any preconditions on the polynomial roots. Secondly, even if we obtain damped oscillations, at least for monthly data, eigenvalues' modulus is near to one so as to generate persistent endogenous fluctuations. In our view, these results are in line with those obtained by Hommes et al. (2005) and Hommes et al. (2008) in a controlled experimental environment for asset pricing. Indeed, they show how realized asset prices exhibit oscillations around the fundamental price with a slow dynamic convergence to it. Lastly, as also stressed in the introduction, most of the empirical nonlinear models do not perform better than a linear chartist-fundamentalist model, and often their fit is worse than the fit of the linear benchmark (Lux, 2021b).

4 Rolling Window Analysis

In an economic environment characterized by the contagion of beliefs, the percentage of agents could vary. To highlight it, we perform a state-space rolling window analysis to check how the percentage of heterogeneous traders changes over time. With this technique, we can assess the dynamics of δ and $1-\delta$ over a rolling window of a fixed size through the sample; if the parameters change during the sample, then the rolling estimates capture this variation. This type of analysis allows us to understand if there is a higher intensity of one behavior with respect to the other in a certain period.

Conventionally, the length of the rolling window is arbitrarily determined and depends on the sample size (Inoue et al., 2017). We use a short rolling window size for data collected in short intervals and a larger size for longer intervals. Following Clark and McCracken (2009), for quarterly data, we choose a rolling window size $m = 40$, i.e., the number of consecutive observations per rolling window, and a rolling window size $m = 120$ and $m = 3650$ for monthly and daily observation respectively (Molodtsova and Papell, 2009). These sizes allow us to implement inference on a relatively large dimension and also to check the period before and after the two major financial crises of the last years, the dot.com crisis (2000) and the global financial crisis (2007/2008). Together with the parameters, we calculate the moving mean of the percentage of agents over a sliding window of length five for quarterly data, length 13 for monthly data, and 365 for daily observation.

The obtained results for quarterly data are shown in Fig. 2. The percentage of speculative position slightly increases during the nineties. This is the period before the dot.com crisis (the internet bubble of the late 1990s). After the explosion of the bubble in the 2000, the percentage decreases with a contemporaneous increase of fundamentalist behavior. From the middle of the year 2000, there is a further increase of this percentage until an inversion of tendency after the crisis of 2007/2008, thus generating a global tendency towards stable oscillatory patterns.

The obtained results for monthly data are shown in Fig. 3. The moving-mean of chartists starts around 40% to rise until the year 1999/2000, falls close to 34% and rises again to about 55% until 2008/2009 and then falls again after the global financial crisis.

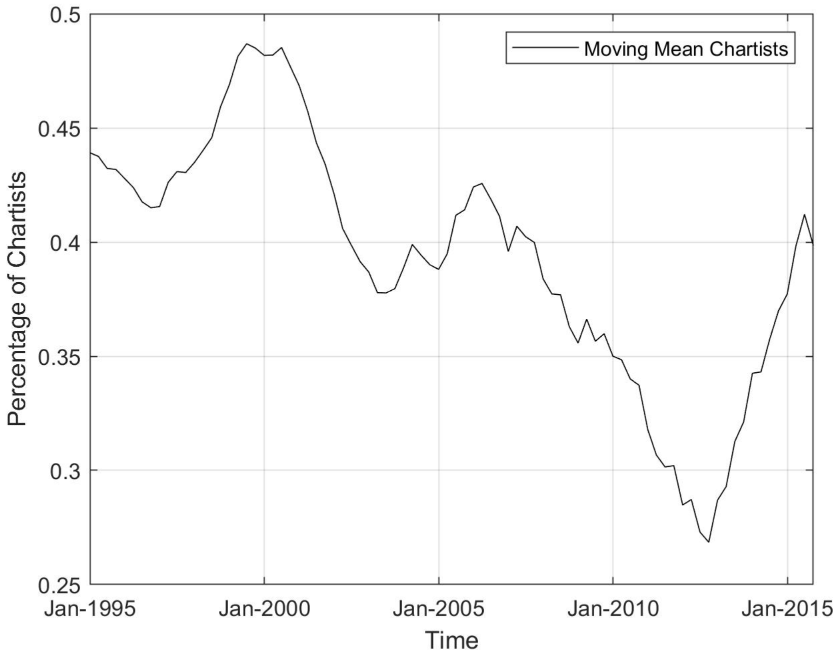

Figure 2: The line represents the dynamics of the moving mean of the parameter $(1-\delta)$ over a length 5 sliding window (Quarterly Data).

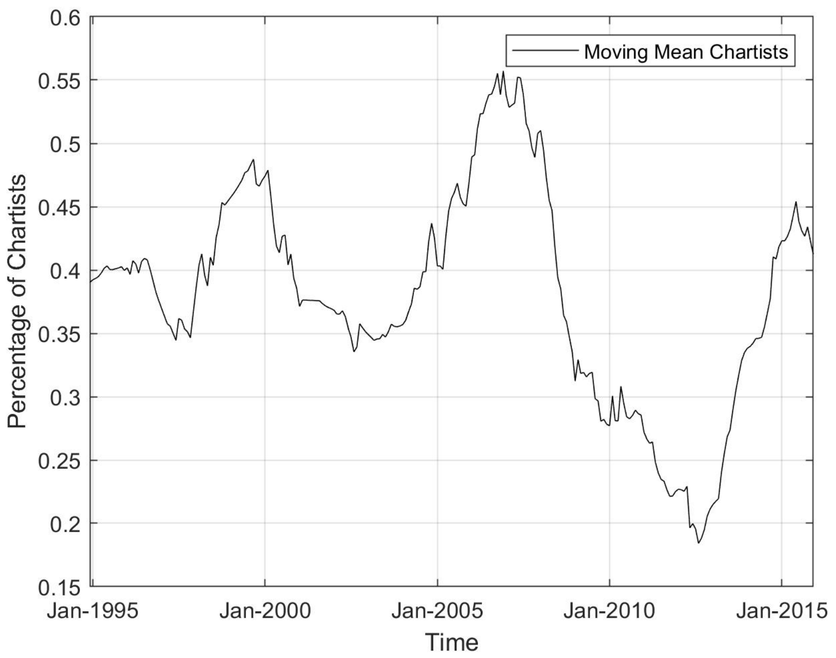

Figure 3: The line represents the dynamics of the moving mean of the parameter $(1-\delta)$ over a length 13 sliding window (Monthly Data).

For daily data, results are shown in Fig. 4. The figure depicts the dynamics of speculative weight and we notice a similar pattern as for the previous cases. The rolling state-space analysis clearly illustrates the effect of a higher intensity of speculative behavior before the year 2000 and before the global financial crisis. The rolling percentage of the moving mean of chartists increases around 85% in 2000, falls after the year 2000, rises again until the end of 2007 ($1-\gamma = 86\%$), and then has a decreasing trend after the year 2008. This implies that speculative behavior reached its peak during the period before the crisis.

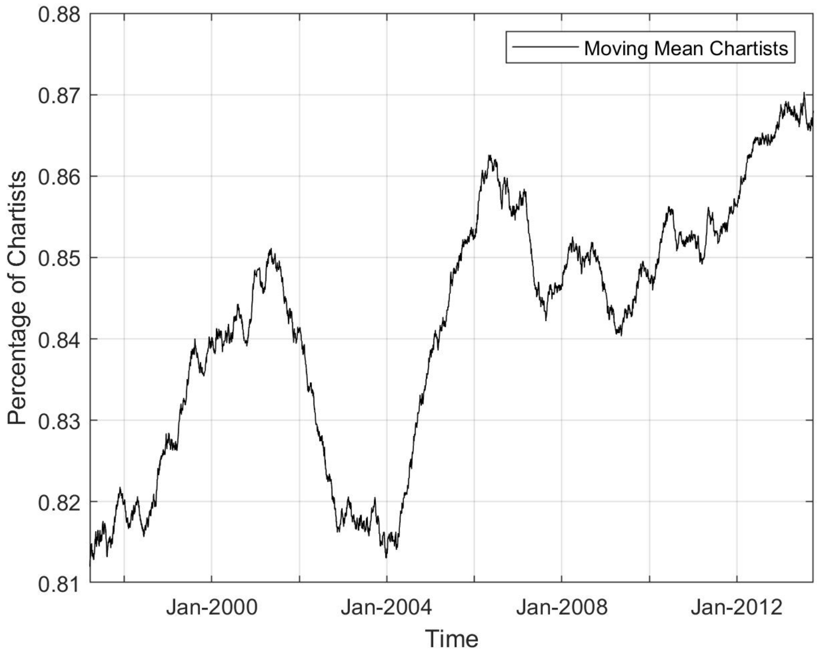

Figure 4: The line represents the dynamics of the moving mean of the parameter $(1-\delta)$ over a length 365 sliding window (Daily Data).

These results, obtained through a state-space rolling window analysis, are consistent with previous results, like Chiarella et al. (2012) and Lof (2012), which showed how fundamental strategies are used more often in times of crisis. In our case, too, we observe an increase in the chartists' behavior before the explosion of the dot.com bubble and the global financial crisis. This means that speculative positions characterize the upward trend while the downward trend has been dominated by the agents adopting a fundamentalist behavior.

5 Out-of-Sample Forecasting Procedure

To complete the analysis, we examine and compare the HA models' performance with that of a benchmark random walk model at various forecast horizons (short vs. long horizon) and different frequency-time (quarterly, monthly, and daily).

We divide our sample (1990-2020) into two segments; the in-sample estimation and the out-of-sample forecast. The out-of-sample period coincides with the last three years (2018-2020) so as to ensure many forecast observations to conduct inference. Concerning the forecasting procedure, the prediction error decomposition approach is repeated for all the out-of-sample period in the forecasting exercise. To perform it, after running the Kalman filter up to time T, we have the starting values for the forecast of the observed series from the current estimate of the state vector (Durbin and Koopman, 2012). In particular, we use the final state predictor $Z_{t\,|\,t}$

together with the measurement and transition equation to construct $P_{T+h|T}$ where $h = 1, 2, ...12$ (for quarterly data), $h = 1, 2, ...36$ (for monthly data) and $h = 1, 2, ...1095$ (for daily data). At the same time, since the sample separation is arbitrary, we perform a state-space rolling window estimation forecasting analysis with a fixed sample size to forecast the h period ahead (see Fig. 5). In this way, we aim to mimic the behavior of financial traders who update their forecasting when new information arrives in each period.

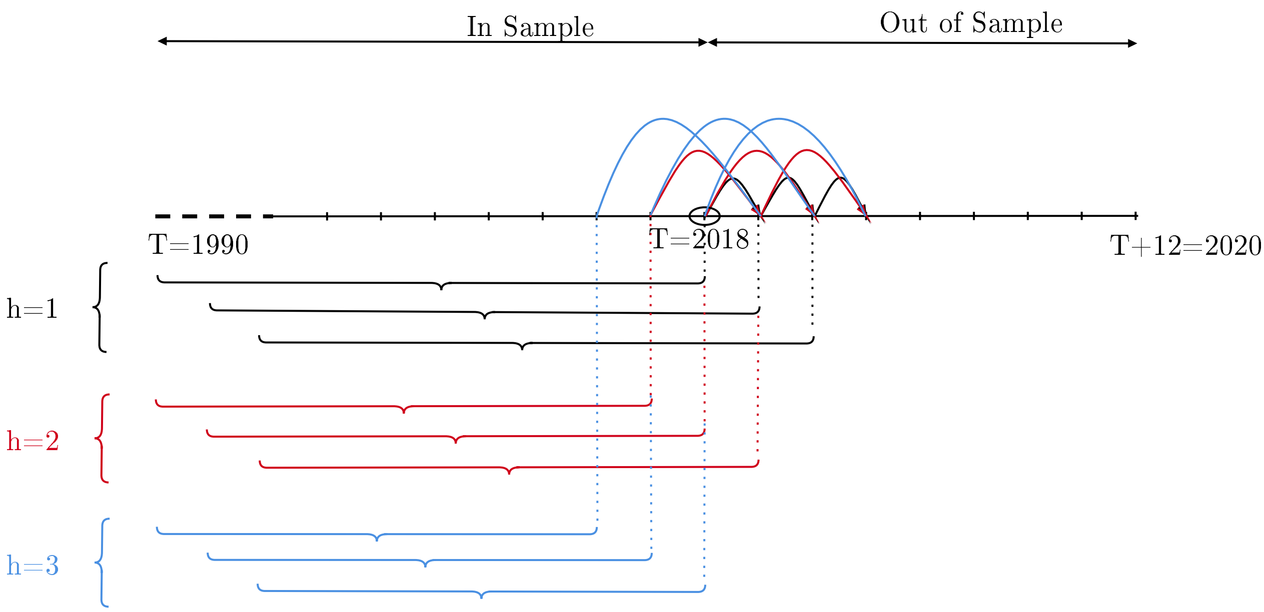

Figure 5: State-space rolling-window forecasting procedure for quarterly data.

We test the robustness of our analysis by looking at each specification through the Mincer and Zarnowitz test (Mincer and Zarnowitz, 1969). In this way, we can evaluate a possible systematic bias in the historical forecasts.[13] Second, we compare the two forecasting performances (HAMs vs random walk) by looking at the ratios of root mean square error (RMSE) and mean absolute error (MAE) for different forecasting horizons. Finally, the Diebold-Mariano test is implemented to determine whether the two specification forecasts are significantly different (Diebold and Mariano, 1995).[14]

Following the Mincer-Zarnowitz test, to have unbiased forecasts, the regression of actual value on the ex-ante forecast should have a zero intercept and a coefficient of one. Table 8 illustrates the obtained results for the behavioral specification. Overall, for all the frequencies considered, coefficients are different from zero at one percent level and values are very close to one, indicating systematic unbiased in the historical forecasts.

The comparison of the out-of-sample forecasting performance based on RMSE and MAE is summarized in Tables 9, 10 and 11. The columns represent the ratio of root mean squared error (RMSE) and the ratio of the mean absolute forecast error (MAE) of the behavioral model (BM) to that of the random walk (RW). A number greater than one represents a better performance of the RW hypothesis, while a value lower than one means a better forecasting performance of the BM specification. h represents the forecast horizons and

[13] On this point, see Appendix C.
[14] On this point, see Appendix D.

Table 8: Mincer-Zarnowitz regression test

	Quarterly Frequency				
h	1 q. h.	3 q. h.	6 q. h.	9 q. h.	12 q. h.
	1.0248***	1.0303***	1.0242***	1.0248***	1.0393***
	Monthly Frequency				
h	1 m. h.	4 m. h.	8 m. h.	12 m. h.	16 m. h.
	1.0249***	1.0224***	0.9879***	1.0171***	1.0264***
h	20 m. h.	24 m. h.	28 m. h.	32 m. h.	36 m. h.
	1.0256***	1.0251***	1.0252***	1.0253***	1.0393***
	Daily Frequency				
h	1 d. h.	365 d. h.	547 d. h.	730 d. h.	1095 d. h.
	1.0026***	1.0025***	1.0025***	1.0026***	1.0393***

Notes: h is the forecasting horizon.
q. h., m. h., d. h. stand for quarterly, monthly and daily horizon respectively.
*, **, *** denotes statistical significance at the 10%, 5%, and 1% levels respectively.

the Diebold-Mariano t-statistics are in parentheses. The two forecasts have the same accuracy under the null hypothesis.

From Table 9, we observe that for quarterly forecasting horizons $\{h = 1, 2, ..., 8\}$, the RMSE is smaller in the RW model than in the behavioral model. Together with this result, the Diebold-Mariano test suggests that the RW model has a better forecasting performance than the heterogeneous behavioral model up to 5 quarterly horizon at one percent level and up to 6 quarterly horizon at five percent level. For h equal to 7 and 8, the two models are statistically indifferent in out-of-sample forecasting power. Conversely, for the forecasting horizons $\{h = 9, ..., 12\}$, RMSE is smaller for the behavioral specification compared to the RW hypothesis with better forecasting power at one percent statical level from $h = 10$. With respect to MAE, we obtain similar results.

To summarize, the Diebold-Mariano test suggests that the two model specifications are statistically different from each other in the short and long periods: the RW is better in the short period, while for a longer period, the heterogeneous behavioral specification outperforms the RW benchmark specification. On this last point, our results are directly in line with those obtained by Jongen et al. (2012) and De Jong et al. (2010).

Table 9: Out-of-sample forecast results (quarterly frequency)

h	1 q. h.	2 q. h.	3 q. h.	4 q. h.	5 q. h.	6 q. h.
$RMSE$	1.9254***	2.5118***	2.6063***	1.4911***	1.4437***	1.2067**
	(-3.30)	(-6.37)	(-5.18)	(-3.42)	(-3.81)	(-2.19)
MAE	2.2138***	2.9914***	2.8445***	1.5800***	1.5187***	1.1991
	(-2.98)	(-6.23)	(-5.13)	(-3.00)	(-4.18)	(-1.62)
h	7 q. h.	8 q. h.	9 q. h.	10 q. h.	11 q. h.	12 q. h.
$RMSE$	1.1167	1.0564	0.8666*	0.8005***	0.7566***	0.7045***
	(-1.60)	(-0.62)	(-1.92)	(-3.25)	(-3.96)	(-5.10)
MAE	1.1667*	1.0678	0.8590**	0.7916***	0.7377***	0.6858***
	(-1.69)	(-0.73)	(-2.22)	(-3.48)	(-4.78)	(-7.47)

Notes: h is the forecasting horizon.
q. h. stands for quarterly horizon.
*, **, *** denotes statistical significance at the 10%, 5%, and 1% levels respectively.

As we did for quarterly data, we now compare the forecasting power of the two model specifications for monthly and daily data. The obtained results confirm the previous results: up to $h = 22$ monthly horizon forecast (Table 10), both the RMSE and the MAE results suggest that RW statistically outperforms the behavioral model. However, it is worth noting a nonsignificant result of the Diebold-Mariano test at the 22 and 24 monthly horizons. Conversely, the HAM exhibits a more accurate forecasting ability between 26 and 36 quarterly horizon. For daily data (Tab. 11), RW statistically outperforms behavioral specification in the short horizon with statistical significance at one percent level, while, in the long horizon, the behavioral model statistically outperforms the random walk.

Our study reveals that the predictive ability of behavioral models depends on the forecast horizon. We believe that the results achieved follow from the intrinsic approach adopted. The RW is strongly linked to the short-trend of the current value of the S&P500; therefore, it fails to capture the long-term changes also because it does not carry relevant information. On the other hand, considering the dividends distributed, the behavioral model with the Gordon specification captures the profit prospects of the medium-long term and, consequently, allows better predictive performance. Overall, this evidence suggests that a model of asset price determination should combine both fundamental and non-fundamental factors (Manzan and Westerhoff, 2007).

Table 10: Out-of-sample forecast results (monthly frequency)

h	2 m. h.	4 m. h.	6 m. h.	8 m. h.	10 m. h.	12 m. h.
$RMSE$	3.9107***	2.0883***	2.1061***	2.0498***	2.0350***	1.8260***
	(-9.77)	(-5.86)	(-8.51)	(-9.17)	(-8.34)	(-7.46)
MAE	4.7643***	2.4895***	2.5597***	2.6038***	2.3257***	1.9929***
	(-14.0)	(-7.72)	(-11.1)	(-10.6)	(-11.9)	(-10.2)
h	14 m. h.	16 m. h.	18 m. h.	20 m. h.	22 m. h.	24 m. h.
$RMSE$	1.6305***	1.8985***	1.2665***	1.1420***	1.0528	0.9656
	(-7.54)	(-6.10)	(-4.64)	(-2.65)	(-1.14)	(-0.66)
MAE	1.7581***	1.4924***	1.3100***	1.1897***	1.0815*	0.9835
	(-9.85)	(-7.37)	(-5.00)	(-3.42)	(-1.68)	(-0.32)
h	26 m. h.	28 m. h.	30 m. h.	32 m. h.	34 m. h.	36 m. h.
$RMSE$	0.9157*	0.8497***	0.7972***	0.7609***	0.7311***	0.5908***
	(-1.71)	(-2.96)	(-4.61)	(-6.00)	(-6.95)	(-8.82)
MAE	0.9054**	0.8454***	0.7861***	0.7427***	0.7069***	0.6668***
	(-1.96)	(-3.33)	(-5.31)	(-6.82)	(-8.03)	(-10.8)

Notes: h is the forecasting horizon.

m. h. stands for monthly horizon.

*, **, *** denotes statistical significance at the 10%, 5%, and 1% levels respectively.

Table 11: Out-of-sample forecast results (daily frequency)

h	91 d. h.	182 d. h.	273 d. h.	365 d. h.	456 d. h.	547 d. h.
$RMSE$	2.1155***	1.7154***	1.4061***	1.1669***	0.9813**	0.8884***
MAE	2.4853***	1.8343***	1.5004***	1.2095***	1.0166***	0.6699***
h	638 d. h.	730 d. h.	821 d. h.	912 d. h.	1003 d. h.	1095 d. h.
$RMSE$	0.7820***	0.6955***	0.6105***	0.5438***	0.4853***	0.4394***
MAE	0.7704***	0.6699***	0.5846***	0.5196***	0.4635***	0.4198***

Notes: h is the forecasting horizon.

d. h. stands for daily horizon.

*, **, *** denotes statistical significance at the 10%, 5%, and 1% levels respectively.

6 Conclusions and Open Questions

The behavioral approach in economics is now mature. It is recognized that economic agents do not form their expectations as rational agents with superior cognitive capacities that cause no bubbles and crashes. These unobservable self-fulfilling beliefs produce waves of optimism and pessimism, making the economy more systemically fragile and possibly giving rise to cycles. In this way, a non-stationary economy must experience at least some fleeting moments of disequilibrium so that observed prices depend on "the state of the market".

This paper proposed a method for estimating possible endogenous cycles in a dynamic model with latent states. Using a simple stylized HAM, we employed a state-space model to analyze the presence of cycles within S&P500. Moreover, we analyzed the model for different periods, specifically on the basis of daily, monthly and quarterly data. At the same time, using state-space rolling windows, we detected changes in the quotas of the two groups of agents. Finally, within this methodology, we compared the out-of-sample predictability of a random walk with the proposed behavioral model.

The obtained results show that the dynamics of asset prices that we see in the data, can be the consequence of our specified behavioral heuristics. In other words, the data reveals the presence of endogenous cycles that can be explained by our HAM.

By focusing on endogenous fluctuations phenomena, we contribute to prior research on the heterogeneity of expectations. With this respect, the use of the state-space models is promising. From a theoretical point of view, this tool allows us to analyze the effect of unobservable components (heuristic decisions) on the dynamics of the observable components (asset prices). From the empirical point of view, the filtering procedure allows us to estimate the unobservable components and recover their associated parameters. More specifically, we can test not only if the interaction of the two main types of expectations governs the dynamics of financial markets but also check if there is empirical evidence of endogenous financial cycles directly from the data.

As future research lines, we believe that a possible way to go deeper into the analysis is to consider the historical series of the agents' expectations. This is beyond the present work but certainly of interest. Using data from BIS, we will be able to compare the evolution of the beliefs extracted from our model with the series that set the agents' expectations. At the same time, we would like to replicate our analysis for other price indices, such as the Nikkei, the Dax or the FTSE. This extension would give the possibility to make a comparison between different financial markets.

We choose to maintain our model as simple as possible due to the substantial number of parameters to estimate and the need to maintain a feasible mathematical condition concerning the search for complex eigenvalues. That said, we want to highlight several possible extensions of the model. On the one hand, because other belief's specifications can generate similar dynamics, it may be valuable to include other strategies adopted, such as contrarian speculators. Moreover, believing that the main problem when considering an asset's fundamental value is the considerable uncertainty in defining it, we could compare different models with different fundamentals. Finally, a Markov switching dynamics in the state-space model formulation could also be included to get local instability together with global instability. All these changes require substantial modifications of the model and of the estimation strategy. We leave this research for future studies.

In conclusion, we would like to quote Cars Hommes: *'The search for a (large) computational agent-based HAM capturing the stylized facts as closely as possible deserves high priority. But at the same time one would like to find the simplest behavioral HAM (e.g. in terms of number of parameters and variables), with a plausible behavioral story at the micro level, that still captures the most important stylized facts observed at the aggregate level'.* In the absence of micro-data on the behavioral strategies, the unobserved components must be extracted by the macro-financial time series. With the use of the state-space methodology, we believe our paper goes in this direction.

Declarations

Competing interests

The authors certify that they have no affiliations with or involvement in any organization or entity with any financial or non-financial interest in the subject matter or materials discussed in this manuscript.

Authors' contributions

Filippo Gusella and Giorgio Ricchiuti contributed equally to the design and implementation of the research, to the analysis of the results and to the writing of the manuscript.

Funding Statement

Filippo Gusella is grateful for financial support from the Università di Firenze and Complexity Lab in Economics (CLE), Università Cattolica del Sacro Cuore, under the joint project "Modelli economico finanziari con agenti eterogenei e learning: approccio teorico e empirico".

Availability of data and materials

The data are available from the authors upon reasonable request.

References

Ahrens, R., Reitz, S., 2005. Heterogeneous expectations in the foreign exchange market, evidence from daily DM/US dollar exchange rates. Journal of Evolutionary Economics 15 (1), 65–82.

Aiyagari, S. R., 1994. Uninsured idiosyncratic risk and aggregate saving. The Quarterly Journal of Economics 109(3), 659-684.

Alfarano, S., Lux, T., Wagner, F., 2006. Estimation of a simple agent-based model of financial markets: An application to Australian stock and foreign exchange data. Physica A: Statistical Mechanics and Its Applications 370(1), 38-42.

Algan, Y., Allais, O., Den Haan, W., Rendahl, P., 2014. Chapter 6 - solving and simulating models with heterogeneous agents and aggregate uncertainty. In Karl Schmedders and Kenneth L. Judd, editors, Handbook of Computational Economics Vol. 3, volume 3 of Handbook of Computational Economics, pages 277- 324.

Bassi, F., Ramos, R., Lang, D., 2023. Bet against the trend and cash in profits: An agent-based model of endogenous fluctuations of exchange rates. Journal of Evolutionary Economics, 1-44.

Beaudry, P., Galizia, D., Portier, F., 2017. Is the Macroeconomy Locally Unstable and Why Should We Care? NBER Macroeconomics Annual 31(1), 479-530.

Beja, A., Goldman, M. B. (1980). On the dynamic behavior of prices in disequilibrium. The Journal of Finance, 35(2), 235–248.

Bewley, T., 1986. Stationary monetary equilibrium with a continuum of independently fluctuating consumers. Contributions to mathematical economics in honor of Gérard Debreu, 79.

Bolt, W., Demertzis, M., Diks, C., Hommes, C., Van Der Leij, M., 2019. Identifying booms and busts in house prices under heterogeneous expectations. Journal of Economic Dynamics and Control, 103, 234-259.

Boswijk, H. P., Hommes, C. H., Manzan, S., 2007. Behavioral heterogeneity in stock prices. Journal of Economic Dynamics and Control 31(6), 1938-1970.

Brock, W. A., Hommes, C. H., 1998. Heterogeneous beliefs and routes to chaos in a simple asset pricing model. Journal of Economic Dynamics and Control 22(8-9), 1235-1274.

Commandeur, J. J. F., Koopman, S. J., 2007. An introduction to state space time series analysis, Oxford: Oxford University Press.

Chen, Z., and Lux, T., 2018. Estimation of sentiment effects in financial markets: A simulated method of moments approach, Computational Economics 52, 711-744.

Chiarella, C., 1992. The Dynamics of speculative behavior. Annals of Operations Research, 37, 101-123.

Chiarella, C., Dieci, R., He, X., 2009. Heterogeneity, market mechanisms, and asset price dynamics, In: Hens, T., Schenk-Hoppé, K. R., (Eds.). Handbooks in Finance, Handbook of Financial Markets: Dynamics and Evolution, North-Holland, 277-344.

Chiarella, C., He, X.-Z., Huang, W., Zheng, H., 2012. Estimating behavioural heterogeneity under regime switching. Journal of Economic Behavior and Organization 83(3), 446-460.

Chiarella, C., He, X.-Z., Zwinkels, R. C. J., 2014. Heterogeneous expectations in asset pricing: Empirical evidence from the S&P500. Journal of Economic Behavior and Organization 105, 1-16.

Clark, T. E., McCracken, M. W., 2009. Improving forecast accuracy by combining recursive and rolling forecast. International Economic Review 50(2), 363-395.

Cutler, D. M., Poterba, J. M., Summers, L. H., 1990. Speculative dynamics and the role of feedback traders. American Economic Review 80(2), 63-68.

Day, R., W. Huang, W., 1990. Bulls, bears and market sheep. Journal of Economic Behavior and Organization 14, 299-329.

De Grauwe, P., Dewachter, H., Embrechts, M., 1993. Exchange Rate Theory-Chaotic Models of Foreign Exchange Markets. Blackwell; Oxford.

De Grauwe, P. and M. Grimaldi, 2006. The exchange rate in behavioral finance framework.

Delli Gatti, D., Fagiolo, G., Gallegati, M., Richiardi, M., Russo, A., 2018. Agent-Based Models in Economics: A Toolkit. Cambridge University Press: Cambridge.

Delli Gatti, D., Grazzini, J., 2020. Rising to the challenge: Bayesian estimation and forecasting techniques for macroeconomic Agent Based Models. Journal of Economic Behavior and Organization 178, 875-902.

De Jong, E., Verschoor, W. F., Zwinkels, R. C., 2009. Behavioural heterogeneity and shift-contagion: Evidence from the Asian crisis. Journal of Economic dynamics and control 33(11), 1929-1944.

De Jong, E., Verschoor, W. F., Zwinkels, R. C., 2010. Heterogeneity of agents and exchange rate dynamics: Evidence from the EMS. Journal of International Money and Finance 29:1652-1669.

Diebold FX, Mariano RS (1995) Comparing predictive accuracy. Journal of Business & Economic Statistics 13:253-63.

Dieci, R., Westerhoff, F. 2012. A simple model of a speculative housing market. Journal of Evolutionary Economics, 22, 303-329.

Dosi, G., Roventini, A. 2019. More is different... and complex! the case for agent-based macroeconomics. Journal of Evolutionary Economics, 29, 1-37.

Dosi, G., Napoletano, M., Roventini, A., Stiglitz, J. E., & Treibich, T., 2020. Rational heuristics? Expectations and behaviors in evolving economies with heterogeneous interacting agents. Economic Inquiry, 58(3), 1487-1516.

Durbin, J., and Koopman, S. J., 2012. Time series analysis by state space methods (Vol. 38), OUP Oxford.

Enders, W., 2016. Applied econometric time series (4th ed.). Supplementary Manual. New York: Wiley.

Fagiolo, G., Guerini, Mattia, L., Francesco, M., A., Roventini, A., 2019. Validation of Agent-Based Models in Economics and Finance. In Claus Beisbart & Nicole J. Saam (eds.), Computer Simulation Validation: Fundamental Concepts, Methodological Frameworks, and Philosophical Perspectives. Springer. pp. 763-787.

Franke, R., 2008. Models of asset price dynamics from Beja-Goldman to Brock-Hommes: A synopsis of specification details. University of Kiel.

Franke, R., 2009. Applying the method of simulated moments to estimate a small agent-based asset pricing model. Journal of Empirical Finance 16(5), 804-815.

Franke, R., Westerhoff, F., 2011. Estimation of a structural stochastic volatility model of asset pricing. Computational Economics 38(1), 53-83.

Franke, R, Westerhoff, F., 2017. Taking stock: rigorous modeling of animal spirits in macroeconomics. Journal of Economic Surveys 31(5), 1152-1182.

Frankel, J. A., Froot, K. A., 1990. Chartists, Fundamentalists and the demand for dollars. In Courakis A.S. and Taylor M. (eds) Private Behavior and Government Policy in Interdependent Economics 73-128.

Frijns, B., Zwinkels, R. C. J., 2018. Time-varying arbitrage and dynamic price discovery. Journal of Economic Dynamics and Control 91, 485-502.

Goldbaum, D., Zwinkels, R. C. J., 2014. An empirical examination of heterogeneity and switching in foreign exchange markets. Journal of Economic Behavior and Organization 107, 667-684.

Gordon, M. J., 1959. Dividends, earnings, and stock prices. The Review of Economics and Statistics 41, 99-105.

Gori, M., Ricchiuti, G. 2018. A dynamic exchange rate model with heterogeneous agents. Journal of Evolutionary Economics, 28, 399-415.

Grazzini, J., Richiardi, M., 2015. Estimation of ergodic agent-based models by simulated minimum distance. Journal of Economic Dynamics and Control 51, 148-165.

Gusella, F., Stockhammer, E., 2021. Testing fundamentalist-momentum trader financial cycles. An empirical analysis via the Kalman filter. Metroeconomica, 72 (2021) 758-797.

Harvey, A. C., 1989. Forecasting, structural time series models and the Kalman filter. Cambridge: Cambridge University Press.

Hommes, C. H., 2006. Heterogeneous Agent Models in Economics and Finance. In: Tesfatsion, L., Judd, K. L. (Eds.). Handbook of Computational Economics, Vol. II. Amsterdam: North-Holland, 1109-1186.

Hommes, C. H., 2021. Behavioral and experimental macroeconomics and policy analysis: A complex systems approach. Journal of Economic Literature 59, 149-219.

Hommes, C., in't Veld, D., 2017. Booms, busts and behavioural heterogeneity in stock prices. Journal of Economic Dynamics and Control 80, 101-124.

Hommes, C., Sonnemans, J., Tuinstra, J., Van de Velden, H., 2005. Coordination of Expectations in Asset Pricing Experiments. The Review of Financial Studies, 18(3), 955-980.

Hommes, C., Sonnemans, J., Tuinstra, J., Van de Velden, H., 2008. Expectations and bubbles in asset pricing experiments. Journal of Economic Behavior and Organization, 67(1), 116-133.

Huggett, M., 1993. The risk-free rate in heterogeneous-agent incomplete-insurance economies. Journal of economic Dynamics and Control 17(5-6), 953 - 969.

Jongen, R., Verschoor, W. F., Wolff, C. C., Zwinkels, R. C., 2012. Explaining dispersion in foreign exchange expectations: A heterogeneous agent approach. Journal of Economic Dynamics and Control 36:719-735.

Inoue, A., Jin, L., Rossi, B., 2017. Rolling window selection for out-of-sample forecasting with time-varying parameters. Journal of Econometrics 196(1), 55-67.

Kirman, A. P., 1992. Whom or What Does the Representative Individual Represent? Journal of Economic Perspectives 6 (2), 117-136.

Krusell, P., Smith, A . A. Jr. Income and wealth heterogeneity in the macroeconomy. Journalof Political Economy, 106(5):867–896, 1998.

Kukacka, J., Barunik, J., 2017. Estimation of financial agent-based models with simulated maximum likelihood. Journal of Economic Dynamics and Control 85, 21-45.

Kukacka J., Sacht S., 2023 Estimation of heuristic switching in behavioral macroeconomic models. Journal of Economic Dynamics and Control, 146, Article 104585

Lamperti, F., Roventini, A., Sani, A., 2018. Agent-based model calibration using machine learning surrogates. Journal of Economic Dynamics and Control, 90, 366-389.

LeBaron, B., 2006. Agent-based Computational Finance. In: Tesfatsion, L., Judd, K. L. (Eds.). Handbook of Computational Economics, Vol. II. Amsterdam: NorthHolland, 1187-1233.

LeBaron, B., Tesfatsion, L., 2008. Modeling macroeconomies as open-ended dynamic systems of interacting Agents. American Economic Review 98(2), 246-50.

Lof, M., 2012. Heterogeneity in stock prices: A STAR model with multivariate transition function. Journal of Economic Dynamics and Control 36(12), 1845-1854.

Lof, M., 2015. Rational speculators, contrarians, and excess volatility. Management Science 61 (8), 1889–1901

Lux, T., 1995. Herd behaviour, bubbles and crashes. The Economic Journal 105(431), 881-896 .

Lux, T., 2018. Estimation of agent-based models using sequential Monte Carlo methods. Journal of Economic Dynamics and Control 91, 391-408.

Lux, T., 2021. Bayesian estimation of agent-based models via adaptive particle Markov chain Monte Carlo. Computational Economics, 1-27.

Lux, T., 2021b. Can heterogeneous agent models explain the alleged mispricing of the S&P 500? Quantitative Finance 21(9), 1413-1433.

Lux, T., Zwinkels, R. C. J., 2018. Empirical validation of agent-based models. Handbook of computational economics 4, 437-488.

Manzan, S., Westerhoff, F., 2007. Heterogeneous expectations, exchange rate dynamics and predictability. Journal of Economic Behavior and Organization 64(1), 111-128.

Mincer, J., Zarnowitz, V. 1969. The Evaluation of Economic Forecasts. In J. Mincer (Ed.), Economic Forecasts and Expectations. New York: National Bureau of Economic Research, 1-46.

Molodtsova, T., Papell, D. H., 2009. Out-of-sample exchange rate predictability with Taylor rule fundamentals. Journal of International Economics 77(2), 167-180.

Monti, C., Pangallo, M., Morales, G. D. F., Bonchi, F., 2022. On learning agent-based models from data. arXiv preprint arXiv:2205.05052.

Poterba, J. M., and Summers, L. H., 1988. Mean reversion in stock prices: Evidence and implications. Journal of financial economics 22(1), 27-59.

Recchioni, M. C., Tedeschi, G., Gallegati, M.A., 2015. A calibration procedure for analyzing stock price dynamics in an agent-based framework. Journal of Economic Dynamics and Control 60, 1-25.

Schmitt, N., 2021. Heterogeneous expectations and asset price dynamics, Macroeconomic Dynamics 25, 1538-1568.

Ter Ellen, S., Verschoor, W., 2018. Heterogeneous beliefs and asset price dynamics: a survey of recent evidence. In: Jawadi, F. (Eds.). Essays in honor of Georges Prat, Springer Nature Switzerland, 53-79.

Ter Ellen, S., Hommes, C., and Zwinkels, R. C. J., 2021. Comparing Behavioural Heterogeneity Across Asset Classes. Journal of Economic Behavior and Organization, 185, 747-769.

Appendix A

In this appendix, eigenvalues analysis is performed to study the conditions for oscillations in our dynamical system. We obtain the associated characteristic equation considering the determinant of the transition matrix A:

$$\begin{vmatrix} \beta(1-\delta)-\lambda & -\beta(1-\delta) \\ 1 & -\lambda \end{vmatrix} = 0,$$

so that:

$$-\lambda(\beta(1-\delta)-\lambda)+\beta(1-\delta)=0,$$

i.e.:

$$\lambda^2-\lambda\beta(1-\delta)+\beta(1-\delta)=0,$$

from which:

$$\lambda_{1,2}=\frac{\beta(1-\delta)\pm\sqrt{(\beta(1-\delta))^2-4\beta(1-\delta)}}{2}.$$

In order to have an oscillating behavior, these last two eigenvalues have to be complex, so we require:

$$\Delta = (\beta(1-\delta))^2 - 4\beta(1-\delta) < 0,$$

i.e.:

$$\beta(1-\delta)(\beta(1-\delta)-4) < 0. \tag{A.1}$$

When this is the case:

$$\lambda_{1,2} = \frac{\beta(1-\delta)}{2} \pm i\frac{\sqrt{-\left[(\beta(1-\delta))^2 - 4\beta(1-\delta)\right]}}{2} = a \pm ib.$$

The complex number in the cartesian form $a \pm ib$ can be written in the equivalent trigonometric form $\rho(\cos\omega \pm i\sin\omega)$.

In order to have oscillations of constant amplitude, we require:

$$\rho = \left(a^2 + b^2\right)^{\frac{1}{2}} = 1,$$

where ρ is the modulus of the complex number.

Solving this simple equation with respect to the parameters of interest, we obtain:

$$\sqrt{\frac{(-\beta(1-\delta))^2}{4} + \frac{-(-\beta(1-\delta))^2 + 4(\beta(1-\delta))}{4}} = 1,$$

i.e.:

$$\sqrt{\beta(1-\delta)} = 1.$$

If Eq. A.1 holds with $\sqrt{\beta(1-\delta)} < 1$ ($\rho < 1$), we observe damped oscillations. With $\sqrt{\beta(1-\delta)} > 1$ ($\rho > 1$), we have explosive oscillations. Finally, if $\sqrt{\beta(1-\delta)} = 1$ ($\rho = 1$), the dynamical system is governed by constant amplitude fluctuations.

Appendix B

In this appendix, we explain the Kalman filtering procedure. The filter gives an algorithm to determine the optimal estimator of the unobserved state vector. The goal is to minimize the mean square prediction error of the unobserved state vector conditional of the observation of $P_t = p_t - p^f$.

The optimal forecasting rule has the form:

$$Z_{t\,|\,t} = Z_{t\,|\,t-1} + K_t\left(P_t - P_{t\,|\,t-1}\right),$$

where K_t is a weight that changes as new information becomes available, $Z_{t\,|\,t}$ denotes the forecast of state variable once P_t is realized while $Z_{t\,|\,t-1}$ and $P_{t\,|\,t-1}$ denote respectively the forecast of variables Z_t and P_t before P_t is realized.

We select the optimal value of K_t to minimize the mean square prediction error at time t:

$$\min_{k_t} E_t\big(Z_t - Z_{t\,|\,t}\big)^2 = \min_{k_t} E_t\big[Z_t - \big(Z_{t\,|\,t-1} + K_t\left(P_t - P_{t\,|\,t-1}\right)\big)\big]^2.$$

In a vector-matrix formulation for the observable variable, we obtain:

$$\min_{k_t} E_t\big[Z_t - \big(Z_{t\,|\,t-1} + K_t\left(HZ_t - HZ_{t\,|\,t-1}\right)\big)\big]^2,$$

from which

$$\min_{k_t} E_t\big[(I - HK_t)\left(Z_t - Z_{t\,|\,t-1}\right)\big]^2,$$

i.e.

$$\min_{k_t} (I - HK_t)^2 E_t\big(Z_t - Z_{t\,|\,t-1}\big)^2.$$

Optimizing with respect to K_t we get:

$$-2H\left(I - HK_t\right) E_t\big(Z_t - Z_{t\,|\,t-1}\big)^2 = 0.$$

Indicating with $\Gamma_{t\,|\,t-1} = E_t\big(Z_t - Z_{t\,|\,t-1}\big)^2$, we obtain:

$$-2H\left(I - HK_t\right)\Gamma_{t\,|\,t-1} = 0.$$

Solving for K_t we obtain:

$$K_t = \frac{H\,\Gamma_{t\,|\,t-1}}{H\,\Gamma_{t\,|\,t-1}H'}.$$

Regrouping the equations, we obtain that:

$$Z_{t\,|\,t-1} = AZ_{t-1\,|\,t-1}, \tag{B.1}$$

$$\Gamma_{t\,|\,t-1} = A\,\Gamma_{t-1\,|\,t-1}A' + Q, \tag{B.2}$$

$$P_{t\,|\,t-1} = HP_{t-1\,|\,t-1}.$$

Equations B.1 and B.2 are the so-called prediction equations. They give the optimal estimates of future values based on the current information set. The other equations are the updating equations:

$$K_t = \Gamma_{t\,|\,t-1} H' (\psi_t)^{-1}, \tag{B.3}$$

with

$$\psi_t = H \Gamma_{t\,|\,t-1} H',$$

$$Z_{t\,|\,t} = Z_{t\,|\,t-1} + K_t \left(P_t - P_{t\,|\,t-1} \right), \tag{B.4}$$

$$\Gamma_{t\,|\,t} = (I - K_t H)\, \Gamma_{t\,|\,t-1}. \tag{B.5}$$

In this case, the inference about Z_t is updated using the observed value of P_t.

We start with a specification information set with initial conditions $Z_{0\,|\,0}$ and $\Gamma_{0\,|\,0}$. Then we use the prediction equations B.1 and B.2 to obtain $Z_{1\,|\,0}$ and $\Gamma_{1\,|\,0}$. Once we observe P_1 we use the updating equations B.3, B.4, and B.5 to obtain $Z_{1\,|\,1}$, $\Gamma_{1\,|\,1}$ and $P_{1\,|\,1}$. We next use this information to form $Z_{2\,|\,1}$ and $\Gamma_{2\,|\,1}$, then forecasts are updated and we continue to repeat this process until the end of the dataset.

Given the vector prediction errors $\mu_t = P_t - P_{t\,|\,t-1}$ and the variance-covariance matrix ψ_t, we can form the log-likelihood to be maximized and to estimate the parameters of interest.

$$\log l = -\tfrac{T}{2} \ln(2\pi) - \tfrac{1}{2} \sum_{t=1}^{T} \ln\left(\left| \psi_{t\,|\,t-1} \right| \right) - \tfrac{1}{2} \sum_{t=1}^{T} \mu_t{}' \left(\psi_{t\,|\,t-1} \right)^{-1} \mu_t \ .$$

Appendix C

The Mincer-Zarnowitz test verifies that forecast errors have zero means and are uncorrelated with any other variable at the time of the forecast. To perform the test for out-of-sample predictions, we run the following regression:

$$p_{T+h} = b_0 + b_1 p_{T+h|T} + \varepsilon_{T+h|T},$$

where p_{T+h} and $p_{T+h|T}$ are the actual and predicted values of asset prices respectively.

In order to have:

$$\varepsilon_{T+h|T} = p_{T+h} - p_{T+h|T},$$

these two conditions should hold:

$$b_0 = 0, \quad b_1 = 1$$

i.e., the regression of the actual value on the ex-ante forecast should have a zero intercept and a coefficient of 1.

Technically, the null hypothesis is:

$$H_0 : b_0 = 0,\ b_1 = 1,$$

against the alternative one:

$$H_1 : b_0 \neq 0,\ b_1 \neq 1.$$

If the null hypothesis is rejected, it indicates systematic bias or inefficiency in the forecasts.

Appendix D

We perform the Diebold-Mariano test to compare the predictive accuracy of different forecasts. We define the forecast errors with the random walk specification (RW) as:

$$\varepsilon^{RW}_{T+h|T} = p_{T+h} - p^{RW}_{T+h|T,}$$

while for the HAM behavioral specification (B) as:

$$\varepsilon^{B}_{T+h|T} = p_{T+h} - p^{B}_{T+h|T.}$$

The loss associated with RW and B forecast is assumed to be a function k of the forecast errors, $k\left(\varepsilon^{RW}_{T+h|T}\right)$ and $k\left(\varepsilon^{B}_{T+h|T}\right)$ respectively. We denote these functions with the squared-error loss and the absolute value. In this way, for the RW model, we obtain:

$$k\left(\varepsilon^{RW}_{T+h|T}\right) = \left(\varepsilon^{RW}_{T+h|T}\right)^2 \quad and \quad k\left(\varepsilon^{RW}_{T+h|T}\right) = \left|\varepsilon^{RW}_{T+h|T}\right|.$$

For the behavioral model, we obtain:

$$k\left(\varepsilon^{B}_{T+h|T}\right) = \left(\varepsilon^{B}_{T+h|T}\right)^2 \quad and \quad k\left(\varepsilon^{B}_{T+h|T}\right) = \left|\varepsilon^{B}_{T+h|T}\right|.$$

The loss differential between the two forecasts is:

$$d_{T+h} = k\left(\varepsilon^{B}_{T+h|T}\right) - k\left(\varepsilon^{RW}_{T+h|T}\right).$$

The null hypothesis states that the B and RW forecasts have the same predictive power:

$$H_0 : E\left(d_{T+h}\right) = 0 \quad \forall\left(T+h\right).$$

In other words, the two forecasts have equal accuracy if the loss differential has zero expectation for all $T+h$. The alternative hypothesis states that the Behavioral and RW forecasts have different levels of accuracy:

$$H_1 : E\left(d_{T+h}\right) \neq 0.$$

Under H_0, the Diebold-Mariano test statistics is:

$$\frac{\bar{d} - u}{\sqrt{\sigma^2/h}} \rightarrow N\left(0, 1\right).$$