Heterogeneous beliefs and routes to chaos in a simple asset pricing model

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- Zeeman 1974;
- Goldman (1980);
- Frankle and Froot (1986, 1990);
- Day and Huang (1990);
- Chiarella (1992);
- Brock and Hommes (1997, 1998);
- Farmer (2002), Joshi (2002);
- and so on...

Main Elements:

- Group of Agents with different strategies (Fundamentalists, Chartists, Noise traders, Bias traders, Naive)
- Fixed quotas vs. Evolutionary switching mechanisms
- Dynamical systems
- Walrasian Auctioner vs. Market Maker

BH1998, 'Heterogeneous beliefs and routes to chaos in a simple asset pricing model', JEDC Vol.22, Issue 8-9

- It is a financial market application of BH1997
- There is bounded rationality
- Agents select their strategy rule based upon recent relative performance
- It is a stylized model, to some extent analytically tractable
- Really good: it is formulated in terms of deviation from a RE benchmark

- Agents can invest in risk free or in a risky asset
- The risk free is perfectly elastically supplied and pays an interest r. While the risky asset pays and uncertain dividend;
- Agents are myopic mean-variance maximizers so the demand per trader of type h for the risky asset is:

$$z_{ht} = \frac{E_{ht}[p_{t+1} + y_{t+1} - (1+r)p_t]}{a\sigma^2}$$

  where E_{ht} are belief of trader type h, and y_{t+1} is the dividend, and a is the risk aversion parameter.
- z^s is the constant supply
- n_{ht} is the quota of group h at time t.

- therefore, the equilibrium between demand and supply is

$$\sum_{h=1}^{H} n_{ht} \frac{E_{ht}[p_{t+1} + y_{t+1} - (1+r)p_t]}{a\sigma^2} = z^s$$

- BH98 assumes that $z^s = 0$, therefore the Walrasian market clearing price satisfies:

$$(1+r)p_t = \sum_{h=1}^{H} n_{ht} E_{ht}[p_{t+1} + y_{t+1}]$$

- in an homogeneous world:

$$p_t^* = \sum_{k=1}^{\infty} \frac{E_t[y_{t+k}]}{(1+r)^k}$$

assuming that the dividend is equal for each period

$$p^* = \sum_{k=1}^{\infty} \frac{\bar{y}}{(1+r)^k} = \frac{\bar{y}}{r}$$

Heterogenous Beliefs:

- in a heterogeneous world traders can deviate from the fundamental price:

$$x_t = p_t - p_t^*$$

- and beliefs are:

$$E_{ht}[p_{t+1}] = E_t[p_{t+1}^*] + f_h(x_{t-1}, ...x_{t-L}) = E_t[p_{t+1}^*] + f_{ht}$$

  where f_h represents a strategy rule.

- therefore

$$(1+r)x_t = \sum_{h=1}^{H} n_{ht} f_{ht}$$

- Evolutionary selection strategies follow a discrete choice approach

$$n_{ht} = \frac{exp(\beta U_{h,t-1})}{\sum^{H}_{h=1} U_{h,t-1}}$$

- in BH the evolutionary fitness is given by the realized excess returns (profits)

$$U_{ht} = (p_t + y_t - Rp_{t-1}) \frac{E_{ht}[p_t + y_t - Rp_{t-1}]}{a\sigma^2} + \omega U_{h,t-1}$$

where ω is a memory parameter. Let's assume that $\omega = 0$.

- therefore:

$$U_{ht} = (x_t - Rx_{t-1}) \frac{f_{h,t-1} - Rx_{t-1}}{a\sigma^2}$$

Let's assume that:

$$f_{ht} = g_h x_{t-1} + b_n$$

where g_h is a trend parameter and b_n a bias parameter. Therefore

1. fundamentalist: $g_h = b_n = 0$
2. trend-followers-chartists: $b_n = 0$
3. biased belief: $g_h = 0$

Why a Third Agent Type?

- The two-agent model (fundamentalists vs. chartists) already generates rich dynamics
- However, empirical evidence suggests that real markets contain **at least three** behaviorally distinct groups:
  - **Fundamentalists**: anchor to fundamental value
  - **Chartists**: extrapolate past price trends
  - **Noise traders**: act on sentiment or biased beliefs
- Adding a third type is **mathematically natural**: the discrete choice framework generalises immediately to H strategies
- It enriches the qualitative dynamics without changing the structure of the model

Discrete Choice with Three Strategies

- The discrete choice equation generalises to $H = 3$:

$$n_{h,t+1} = \frac{\delta\, n_{h,t} + (1-\delta)\exp(\beta U_{h,t})}{\sum_{k=1}^{3} \exp(\beta U_{k,t})}, \qquad h = 1, 2, 3$$

where $\delta \in [0,1]$ is an inertia parameter and β is the intensity of choice.

- The denominator is now a sum of **three** exponentials:

$$Z_t = \exp(\beta U_{1,t}) + \exp(\beta U_{2,t}) + \exp(\beta U_{3,t})$$

- The market-clearing condition becomes:

$$(1+r)\, x_t = n_{1,t}\, f_{1,t} + n_{2,t}\, f_{2,t} + n_{3,t}\, f_{3,t}$$

- Shares sum to one by construction: $n_{3,t} = 1 - n_{1,t} - n_{2,t}$

The Three Belief Rules

Each type uses a linear rule $f_{ht} = g_h x_{t-1} + b_h$:

Type	g_h	b_h	**Cost**	**Interpretation**
1 Fundamentalists	0	0	$C_1 > 0$	Return to fundamental
2 Chartists	$g_2 > 1$	0	0	Trend extrapolation
3 Noise traders	free	$b_3 \neq 0$	0	Sentiment / bias

- The bias b_3 acts as a **persistent upward or downward pressure** on the price
- Noise traders pay no information cost but their beliefs are systematically incorrect

Fitness and Profits

- Realized profits for each type (normalising $a\sigma^2 = 1$):

$$\pi_{h,t} = (x_t - R\,x_{t-1})(f_{h,t-1} - R\,x_{t-1}) - C_h$$

- Fitness with memory parameter ω:

$$U_{h,t} = (1-\omega)\,\pi_{h,t} + \omega\,U_{h,t-1}$$

- With $\omega = 0$, fitness coincides with current-period profits
- Agents **switch** toward the strategy that performed best recently this is the key feedback mechanism that drives complex dynamics

The Role of β: Intensity of Choice

- β governs how **sensitive** agents are to performance differences across strategies
- $\beta \to 0$: agents are indifferent — shares stay fixed at $1/3$ each
- $\beta \to \infty$: all agents instantly switch to the best-performing strategy
- For intermediate values of β, the model undergoes a **sequence of bifurcations**:

$$\text{stable fixed point} \longrightarrow \text{2-cycle} \longrightarrow \text{4-cycle} \longrightarrow \cdots \longrightarrow \text{chaos}$$

- This *period-doubling route to chaos* is a classical result — the third agent type can **shift the bifurcation threshold** relative to the two-agent model

The Role of g_2: Chartist Aggressiveness

- g_2 measures how strongly chartists extrapolate past price deviations
- There exists a **critical threshold** g_2^* above which the fundamental steady state loses stability
- Below g_2^*: fundamentalists dominate, price converges to fundamental value
- Above g_2^*: chartists destabilise the market — cycles and chaos emerge
- The presence of noise traders (type 3) **interacts** with this threshold:
  - A positive bias $b_3 > 0$ shifts the attractor upward
  - A sufficiently large $|b_3|$ can trigger instability even at low g_2

What the Three-Agent Model Adds

- **Richer dynamics**: the interaction among three strategies expands the set of possible attractors (fixed points, cycles, chaos, intermittency)
- **More realistic**: noise traders and biased beliefs are well-documented empirically (De Long et al. 1990, Shiller 2000)
- **Asymmetric dynamics**: a positive b_3 can generate *bubbles and crashes* — slow price run-ups followed by sharp reversals
- **Volatility clustering**: with memory ($\omega > 0$), the model can produce episodes of high and low volatility, consistent with financial stylised facts
- The core mechanism — *evolutionary switching driven by past profits* — is unchanged

Open Questions

- What happens if all three types have zero information cost? Does the fundamental equilibrium survive?
- Can the model generate **volatility clustering** (ARCH/GARCH effects) endogenously?
- How sensitive are the dynamics to the **initial shares** (n_{10}, n_{20})?
- How could β and g_2 be **estimated** from financial data?
- Extensions: continuous time, stochastic noise, learning, more than three types

Zero Information Costs: Does the Fundamental Survive?

- If $C_1 = C_2 = C_3 = 0$, fundamentalists lose their only disadvantage
- The discrete choice then depends **purely on past profits**
- At the fundamental steady state $x^* = 0$, all profits are zero — shares remain constant at whatever initial value (n_{10}, n_{20}, n_{30})
- However, the steady state may be **unstable**: if chartists earn higher profits off equilibrium, they attract more agents and amplify deviations
- Result: the fundamental equilibrium *can* survive, but only if g_2 is below the destabilisation threshold — the bias b_3 makes it harder

Volatility Clustering

- Volatility clustering: large price changes tend to be followed by large changes (of either sign) — a key stylised fact of financial returns
- In the BH framework it arises **endogenously** when $\omega > 0$:
  - After a large deviation, chartists earn high profits
  - Their share $n_{2,t}$ rises — the market becomes more trend-following
  - This sustains high volatility for several periods
  - Eventually fundamentalists recover and volatility subsides
- With $\omega = 0$ (no memory) the feedback is weaker and clustering is less pronounced
- Formal test: significant ARCH effects in simulated returns V_t confirm the presence of volatility clustering

Sensitivity to Initial Shares

- Initial shares (n_{10}, n_{20}, n_{30}) affect the **transient** path but not the long-run attractor — for most parameter configurations
- Exception: when **multiple attractors** coexist, initial shares determine which basin of attraction the system falls into
- Practical implication: two economies with identical parameters but different initial compositions can converge to very different long-run dynamics
- This is a form of **path dependence** — history matters
- Sensitivity is stronger when β is large (agents switch fast) and δ is small (little inertia)

Estimating β and g_2 from Data

- Direct estimation is challenging: agent shares n_{ht} are **latent variables** — not directly observable
- Main approaches in the literature:
  - **Simulated Method of Moments**: match simulated moments (autocorrelations, kurtosis, volatility) to empirical counterparts
  - **Maximum Likelihood** (Manzan & Westerhoff 2005): write the likelihood conditional on the latent shares
  - **Markov-switching models**: approximate switching between regimes as a proxy for endogenous switching
- Estimated β values in the literature are typically in the range $[1, 5]$ — consistent with the parameter values used in our simulations